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LETTER TO THE EDITOR

Conformal off-diagonal boundary density profiles on a semi-infinite strip

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Abstract
The off-diagonal profile $\phi_{\rm od}^{\rm b}(v)$ associated with a local operator $\hat{\phi}(v)$ (order parameter or energy density) close to the boundary of a semi-infinite strip with width L is obtained at criticality using conformal methods. It involves the surface exponent $x_\phi^{\rm s}$ and displays a simple universal behaviour which crosses over from surface finite-size scaling when v/L is held constant to corner finite-size scaling when $v/L \to 0$.

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The finite-size behaviour of order parameter or energy density profiles has been the subject of much interest during the last two decades following the work of Fisher and de Gennes [1]. These profiles have been studied in the vicinity of the critical point in the mean-field approximation [2], using field-theoretical methods [3] and through exact solutions [4,5]. Such profiles display universal behaviour at criticality and in two-dimensional systems they can be deduced from ordinary scaling and covariance under conformal transformation [6–17]. A short review can be found in [18].

With symmetry-breaking boundary conditions, one may consider diagonal order parameter profiles [6], i.e. ground-state expectation values. Otherwise, off-diagonal profiles can be used with any type of boundary condition [13]. For the order parameter with Dirichlet boundary conditions, off-diagonal matrix elements must be considered since a diagonal order parameter profile then vanishes for symmetry reasons.

On a strip with fixed boundary conditions at $v = 0$ and L the diagonal order-parameter profile $\phi(v)$ associated with an operator $\hat{\phi}$ takes the following form at criticality [6]:

$$\phi(v) = \langle 0|\hat{\phi}(v)|0\rangle = \mathcal{A}\left[\frac{L}{\pi}\sin\left(\frac{\pi v}{L}\right)\right]^{-x_\phi} \qquad 0 < v < L. \tag{1}$$

The exponent x_ϕ is the bulk scaling dimension of $\hat{\phi}$; $|0\rangle$ is the ground state of the Hamiltonian $\mathcal{H} = -\ln\mathcal{T}$ where $\mathcal{T}$ denotes the row-to-row transfer operator on the strip. When $L \to \infty$ with a fixed value of the ratio v/L, one obtains the bulk finite-size scaling behaviour $\phi(L) \sim L^{-x_\phi}$.

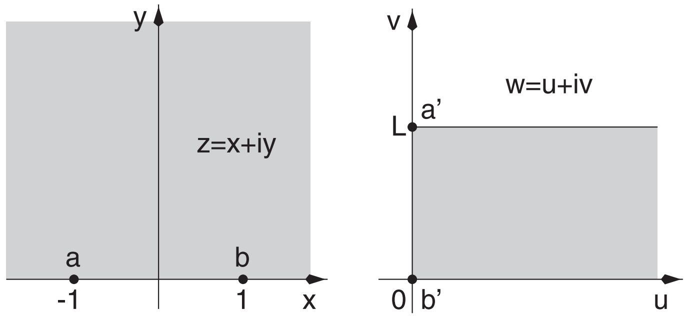

Figure 1. Conformal mapping of the half-plane $y > 0$ on the semi-infinite strip $u > 0, 0 < v < L$.

When $L \to \infty$ while keeping v fixed, one obtains the profile $\phi(v) \sim v^{-x_\phi}$ on the half-plane with fixed boundary conditions, which is a consequence of ordinary scaling. Actually the profile on the strip in (1) follows from the profile on the half-plane through the logarithmic conformal transformation $w = (L/\pi)\ln z$ [6].

The off-diagonal critical profile with general symmetric boundary conditions at $v = 0$ and L is obtained as [13]

$$\phi_{\rm od}(v) = \langle\phi|\hat{\phi}(v)|0\rangle \sim \left(\frac{L}{\pi}\right)^{-x_\phi}\left[\sin\left(\frac{\pi v}{L}\right)\right]^{x_\phi^{\rm s}-x_\phi} \qquad 0 < v < L \tag{2}$$

where $|\phi\rangle$ is the lowest excited state of $\mathcal{H}$ leading to a non-vanishing matrix element. Besides the bulk exponent x_ϕ the off-diagonal profile involves the surface scaling dimension $x_\phi^{\rm s}$ of the operator $\hat{\phi}$. It can be identified by considering the transformation of the connected two-point correlation function $\mathcal{G}_{\phi\phi}^{\rm con}(z_1, z_2)$ from the half-plane to the strip under the logarithmic conformal mapping. For the order parameter with fixed boundary conditions, $x_\phi^{\rm s} = 0$, and (2) gives an off-diagonal profile in agreement with (1). When $L \to \infty$, equation (2) shows the crossover from bulk finite-size scaling $\phi_{\rm od}(L) \sim L^{-x_\phi}$ when the ratio v/L is constant, to surface finite-size scaling $\phi_{\rm od}(L) \sim L^{-x_\phi^{\rm s}}$ when v is constant, i.e. when $v/L \to 0$.

Let us now consider a half-strip in the (u, v)-plane with $0 < u < \infty$, $0 < v < L$ and uniform boundary conditions. If one crosses the semi-infinite strip at $u \gg L$ the behaviour of the off-diagonal profile will be the same as for the infinite strip in equation (2). A different behaviour is expected close to the boundary of the semi-infinite strip at $u \ll L$. The profile should then involve the surface exponent $x_\phi^{\rm s}$ and, when $v \ll L$ or $L - v \ll L$, the corner exponent $x_\phi^{\rm c}$.

In order to obtain the profiles on the semi-infinite strip, we consider the transformation of the connected two-point correlation function $\mathcal{G}_{\phi\phi}^{\rm con}(z_1, z_2)$ from the half-plane $z = x + {\rm i}y$, $y > 0$ to the half-strip $w = u + {\rm i}v$, $0 < u < \infty$, $0 < v < L$ as shown in figure 1. The two geometries are related by the conformal transformation [6]

$$z = \cosh\left(\frac{\pi w}{L}\right) \tag{3}$$

or

$$x = \cosh\left(\frac{\pi u}{L}\right)\cos\left(\frac{\pi v}{L}\right) \qquad y = \sinh\left(\frac{\pi u}{L}\right)\sin\left(\frac{\pi v}{L}\right). \tag{4}$$

Going from the half-plane to the half-strip, the dilatation factor is given by

$$b(z) = \left|\frac{{\rm d}z}{{\rm d}w}\right| = \left|\frac{\pi}{L}\sinh\left(\frac{\pi w}{L}\right)\right| = \frac{\pi}{L}\left[\sinh^2\left(\frac{\pi u}{L}\right) + \sin^2\left(\frac{\pi v}{L}\right)\right]^{1/2}. \tag{5}$$

At criticality, the form of $\mathcal{G}^{\rm con}_{\phi\phi}(z_1, z_2)$ is strongly constrained by conformal invariance. Using an infinitesimal special conformal transformation which preserves the surface geometry, one obtains a system of partial differential equations for the connected two-point correlation function on the half-plane, from which the following scaling form is deduced [19]:

$$\mathcal{G}^{\rm con}_{\phi\phi}(z_1, z_2) = (y_1 y_2)^{-x_\phi} g(\omega) \qquad \omega = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}. \tag{6}$$

We are mainly interested in the behaviour of the profile close to the boundary of the half-strip. Thus we may consider the correlations between two points, the first close to the boundary located at $u = 0$ and the second far from it:

$$\frac{u_1}{L} \ll 1 \qquad 0 < v_1 < L \qquad \frac{u_2}{L} \gg 1 \qquad 0 < v_2 < L. \tag{7}$$

Considering (4) and (5) in the limits of equation (7) we have

$$\begin{aligned} &x_2 - x_1 \simeq \frac{1}{2}\exp\left(\frac{\pi u_2}{L}\right)\cos\left(\frac{\pi v_2}{L}\right) \qquad y_2 - y_1 \simeq \frac{1}{2}\exp\left(\frac{\pi u_2}{L}\right)\sin\left(\frac{\pi v_2}{L}\right) \\ &y_1 y_2 \simeq \frac{\pi u_1}{2L}\sin\left(\frac{\pi v_1}{L}\right)\sin\left(\frac{\pi v_2}{L}\right)\exp\left(\frac{\pi u_2}{L}\right) \\ &b(z_1) \simeq \frac{\pi}{L}\sin\left(\frac{\pi v_1}{L}\right) \qquad b(z_2) \simeq \frac{\pi}{L}\exp\left(\frac{\pi u_2}{L}\right). \end{aligned} \tag{8}$$

Thus the crossover variable ω defined in (6) takes the form

$$\omega \simeq \frac{L\exp(\pi u_2/L)}{2\pi u_1 \sin(\pi v_1/L)\sin(\pi v_2/L)} \gg 1. \tag{9}$$

In this limit, ordinary scaling leads to $g(\omega) \sim \omega^{-x^{\rm s}_\phi}$ so that, in the half-plane geometry,

$$\mathcal{G}^{\rm con}_{\phi\phi}(z_1, z_2) \sim \frac{(y_1 y_2)^{x^{\rm s}_\phi - x_\phi}}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{x^{\rm s}_\phi}}. \tag{10}$$

The conformal mapping (3) leads to the correlation function in the half-strip geometry [20]

$$\mathcal{G}^{\rm con}_{\phi\phi}(w_1, w_2) \sim b(z_1)^{x_\phi} b(z_2)^{x_\phi} \mathcal{G}^{\rm con}_{\phi\phi}(z_1, z_2). \tag{11}$$

Making use of (8) in equations (10) and (11), we finally obtain

$$\mathcal{G}^{\rm con}_{\phi\phi}(w_1, w_2) \sim u_1^{x^{\rm s}_\phi - x_\phi}\left[\left(\frac{2\pi}{L}\right)\sin\left(\frac{\pi v_1}{L}\right)\right]^{x^{\rm s}_\phi}\left(\frac{L}{\pi}\right)^{-x_\phi}\left[\sin\left(\frac{\pi v_2}{L}\right)\right]^{x^{\rm s}_\phi - x_\phi}\exp\left(-\frac{\pi x^{\rm s}_\phi u_2}{L}\right). \tag{12}$$

In order to identify the different contributions to the two-point correlation function in (12), we can rewrite it using the row-to-row transfer operator $\mathcal{T}$ on the strip with width L. The two-point correlation function on the semi-infinite strip reads

$$\mathcal{G}_{\phi\phi}(w_1, w_2) = \frac{\langle B|\hat\phi(v_1)\mathcal{T}^{u_2}\hat\phi(v_2)|0\rangle}{\langle B|\mathcal{T}^{u_2}|0\rangle} = \frac{\sum_n \langle B|\hat\phi(v_1)|n\rangle \exp(-u_2 E_n)\langle n|\hat\phi(v_2)|0\rangle}{\langle B|0\rangle \exp(-u_2 E_0)} \tag{13}$$

where $|0\rangle$ is the ground state of $\mathcal{H}$ which is selected by the transfer from $u = u_2$ to $u = \infty$ and $|B\rangle$ is a state vector appropriate for the boundary conditions at $u = 0$. In the case of free boundary conditions, it describes the free summation over the boundary states. In the last expression the summation is over the complete set of eigenstates $|n\rangle$ of $\mathcal{H}$ with eigenvalues E_n.

The connected two-point correlation function is then obtained by subtracting the ground-state contribution to the eigenstate expansion:

$$\begin{aligned} \mathcal{G}^{\rm con}_{\phi\phi}(w_1, w_2) &= \mathcal{G}_{\phi\phi}(w_1, w_2) - \frac{\langle B|\hat\phi(v_1)|0\rangle}{\langle B|0\rangle}\langle 0|\hat\phi(v_2)|0\rangle \\ &\simeq \frac{\langle B|\hat\phi(v_1)|\phi\rangle}{\langle B|0\rangle}\langle \phi|\hat\phi(v_2)|0\rangle \exp[-u_2(E_\phi - E_0)]. \end{aligned} \tag{14}$$

In the last expression we took into account the condition $u_2 \gg L$. In this limit, the eigenstate expansion is dominated by the contribution of the lowest excited state $|\phi\rangle$ of $\mathcal{H}$ for which the matrix elements are non-vanishing.

Comparing with the conformal expression in (12), we can read the gap-exponent relation [20] in the exponential factor, $E_\phi - E_0 = \pi x_\phi^{\rm s}/L$, and the off-diagonal profile at $u_2 \gg L$ in agreement with equation (2). The remaining part can be then identified as the boundary profile on the half-strip at $u_1 \ll L$ and we obtain

$$\phi_{\rm od}^{\rm b}(v) = \frac{\langle B|\hat{\phi}(v)|\phi\rangle}{\langle B|0\rangle} \sim \left[\frac{\pi}{L}\sin\left(\frac{\pi v}{L}\right)\right]^{x_\phi^{\rm s}} \qquad 0 < v < L. \tag{15}$$

In the case of the two-dimensional Ising model with free boundary conditions $\mathcal{H}$ is the Hamiltonian of the Ising model in a transverse field; if one associates the Pauli spin operator σ_l^x $(l = 1, L)$ with the order parameter, then the boundary state vector $|B\rangle$ is explicitly given by [21]

$$|B\rangle = \prod_{l=1,L} \frac{1}{\sqrt{2}}(|\sigma_l^x = +1\rangle + |\sigma_l^x = -1\rangle) = \prod_{l=1,L} |\sigma_l^z = +1\rangle. \tag{16}$$

Both $|0\rangle$ and $|B\rangle$ are even under the operator $P = \prod_{l=1,L} \sigma_l^z$ [21]. In the expression of the order parameter profile the state $|\phi\rangle = |\sigma\rangle$, which contains a single fermionic excitation, is odd under P. At $l = 1$, the order parameter profile coincides with the corner magnetization obtained in [21]. The surface magnetic exponent is $x_\sigma^{\rm s} = 1/2$ [22]. For the energy density profile the state $|\phi\rangle = |\epsilon\rangle$ contains two fermionic excitations and is even under P. The surface energy exponent is then $x_\epsilon^{\rm s} = 2$ [23, 24].

When $L \to \infty$ with a fixed v/L value, one obtains the surface finite-size scaling behaviour $\phi_{\rm od}^{\rm b}(L) \sim L^{-x_\phi^{\rm s}}$ while keeping v constant leads to the corner finite-size scaling behaviour

$$\phi_{\rm od}^{\rm b}(L) \sim v^{x_\phi^{\rm s}} L^{-2x_\phi^{\rm s}} \qquad v \ll L. \tag{17}$$

Thus the corner exponent $x_\phi^{\rm c}(\pi/2)$ is given by $2x_\phi^{\rm s}$. This result is in agreement with the general expression $x_\phi^{\rm c}(\theta) = \pi x_\phi^{\rm s}/\theta$ for a corner with opening angle θ, which also follows from conformal invariance [19, 21].

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